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Nonstandard Extension of Quantum Logic and Dirac's Bra-Ket Formalism of Quantum Mechanics

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An extension of the quantum logical approach to the axiomatization of quantum mechanics using *nonstandard analysis* methods is proposed. The physical meaning of a quantum logic as a lattice of propositions is conserved by its nonstandard extension. But not only the usual Hilbert space formalism of quantum mechanics can be derived from the nonstandard extended quantum logic. Also the Dirac bra-ket quantum mechanics can be derived as a consequence of such an extended quantum logic.

1. INTRODUCTION

It was shown during the 1960s and 1970s that the Hilbert space formalism of quantum mechanics can be derived from a set of simple axioms of the most general nature. The lattice structures appear, where each physical system can be associated with a partially ordered σ-orthocomplemented set $\mathscr{L}$. Each observable can be identified with an $\mathscr{L}$-valued measure on the real Borel sets and each state can be identified with a probability measure on $\mathscr{L}$. It was shown that there exists a set of axioms which implies that $\mathscr{L}$ is isomorphic to the lattice of all closed subspaces of a complex Hilbert space (Mackey, 1963; Maçzynski, 1972; Piron, 1976; Beltrametti and Casinelli, 1976). Then the observables are nothing else but self-adjoint operators acting on Hilbert space, and the states are rays lying in Hilbert space.

The problem arises when one tries to apply this way of axiomatization to *Dirac's formalism* of quantum mechanics. Dirac's bra-ket formalism is not a separable Hilbert space theory from the mathematical point of view, because it treats on almost the same footing observables with discrete and

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continuous spectra (Dirac, 1958). Eigenstates of these two kinds of observables both exist in the Dirac unitary space of states, the Dirac δ-function, for example, serving as a coordinate observable eigenstate. This space of states cannot therefore be a separable Hilbert one, which is also explicitly shown in Section 4.

On the other hand, the mathematical essence of Dirac's formalism of quantum mechanics was determined to be of the theory of distributions (Roberts, 1966; Melsheimer, 1972). Especially the spectral theory of operators in rigged Hilbert spaces proposed by Gelfand was developed to describe essential properties of the bra-ket formalism (Gelfand and Vilenkin, 1961; Roberts, 1966). In these theories one *can* describe states like the δ-function in a mathematically consistent way as functionals on the space of test functions, where the embedding takes place: the *test function space* $\hat{\mathscr{G}}$ lies within the Hilbert space $\mathbb{H}$, which in its turn lies within space of *functionals* $\hat{\mathscr{G}}'$ on $\hat{\mathscr{G}}$. Then some states of a physical system can be represented as objects that do not belong to the Hilbert space $\mathbb{H}$.

That obviously causes trouble for the quantum logic approach, because in its usual form [for review see Beltrametti and Casinelli (1976)] it leads to the essentially *Hilbertian* structure of the space of states. In the separable Hilbert space formalism that one can get from canonical axiomatic schemes of Mackey (1963) or Piron (1976) (see also Maçzynski, 1972) there are no eigenstates coinciding with points of a *continuous* spectrum of self-adjoint operators acting on the Hilbert space $\mathbb{H}$. There is no place for the δ-function in $\mathbb{H}$ for this particular example of an "eigenvector."

So, an attractive logical structure of quantum axiomatics has a very serious lack—it does not have as a consequence the 'calculation background' of quantum mechanics—the Dirac bra-ket formalism.

Then, a natural question is: Is it possible to modify in some way the usual quantum logic scheme, conserving its logical structure of the lattice of propositions, to get *Dirac's* formalism of quantum mechanics?

In the present paper we propose to build an extension of the quantum logical axiomatic scheme using *nonstandard analysis* methods (Davis, 1958; Stroyan and Luxemburg, 1976; Albeverio *et al.*, 1986).

The nonstandard analysis invented in the 1960s by A. Robinson is a new approach to the infinite and infinitesimal numbers which allows us to treat the finite, infinite, and infinitesimal numbers on the same footing as members of an extended field of numbers $^*\mathbb{R}$, which is often called the *hyperreal* field of numbers. In distinction from the usual real field $\mathbb{R}$ that does not contain numbers which are less or bigger than *any* real number, such curious (for the usual mathematics) objects can be defined in $^*\mathbb{R}$ in a natural and self-consistent way.

Moreover, nonstandard analysis is a very powerful tool for the extension of mathematical models which can be defined as some axiom systems applied to sets of elementary objects—'points,' and not only to the fields of numbers. Thus, in principle, one can apply it for the extension of quantum logic, because it is exactly such a mathematical object.

Of course, also the Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ itself can be extended together with the operators on it to get an extended mathematical model of quantum mechanics. It was shown by Farrukh (1975) that such a "hyper-Hilbert" space $^*\mathbb{H}$ over the field of "hypercomplex" numbers $^*\mathbb{C}$ fits to the Dirac space of states. But to our knowledge no attempt had been made for continuation of this work, especially for the axiomatization of this theory.

In this work, which can be considered as an attempt to build an axiomatic background for Dirac's non-relativistic quantum mechanics, it is shown that in distinction with the usual quantum logic, its nonstandard extension *can* have as a consequence Dirac's formalism of quantum mechanics. That follows from the fact that in this theory spectral properties of observables with discrete and continuous spectra are considered equally. Together with this, the nonstandard extension conserves the logical structure of quantum axiomatics.

This article is built in the following way. In Section 2 the methods of nonstandard analysis are outlined. In Section 3 the quantum logic axioms are formulated in the form needed for its nonstandard extension. One can skip these two sections for review, and they do not contain essentially new results. Section 4 discusses the spectral properties of observables of the usual quantum logic. It is shown that the spectral properties of the observables with discrete and continuous spectra are of very distinct features. "Eigenstates" in their usual mathematical meaning can exist only for the first ones, but not for the second ones. In Section 5 the nonstandard extension of the quantum logic described in Section 3 is built. Finally, Section 6 discusses the spectral properties of observables of the nonstandard quantum logic. A special "spectral axiom" is introduced, which states that the "infinitesimal measuring error does not lead to the change of proposition." This axiom implies the essentially Dirac bra-ket properties of the set of states associated with the nonstandard quantum logic.

2. A SHORT DESCRIPTION OF THE NONSTANDARD ANALYSIS METHODS

The nonstandard analysis operates with such notions as individuals (or points), ultrafilters, superstructures, and languages and the main tool we shall use is the *Transfer Theorem*. We shall give all the necessary definitions here. For more detailed explanations one can turn to Davis (1977) and Albeverio *et al.* (1986).

We shall say that S is a set of *individuals* (or *points*) if every member of S does not contain subsets. Every point is in some sense an irreducible object. One can choose as a set of individuals, for example, Hilbert space $\mathbb{H}$, fields of numbers $\mathbb{R}$ or $\mathbb{C}$, or lattice $\mathscr{L}$.

The *standard superstructure* is built under the set of individuals S as follows:

$$\begin{aligned} V_1(S) &= S \\ V_{i+1}(S) &= V_i(S) \cup \mathscr{P}(V_i(S)) \\ \hat{V}(S) &= \bigcup_{i=1}^{\infty} V_i(S) \end{aligned} \tag{2.1}$$

Here $\mathscr{P}(A) = \{B|B \subseteq A\}$—the power set of A.

The superstructure $\hat{V}(S)$ includes all the mathematical notions we need (individuals, sets, functions on individuals and on sets, Cartesian products of sets, etc.). The notions $x \in A$, $x \subseteq A$, $x \subset A$, $x = y$, $\{x|\cdots\}$, $\{x \in A|\cdots\}$, $A \backslash B$ have their usual set-theoretical meaning in $\hat{V}(S)$. Through $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$ an *ordered pair* of elements of $\hat{V}(S)$ is denoted.

For two sets A and B their *Cartesian product* is defined as $A \times B = \{\langle a, b\rangle | x \in A, y \in B\}$. If $R \subseteq A \times B$ is a *relation*, then we write xRy in place of $\langle x, y\rangle \in R$. By dom($R$) the domain of relation R is denoted.

By $\{X_i | i \in I\}$ a function from the set I to the set X is denoted when we are more interested in its range than in its domain. The set I is called then an *index set*, and instead of a function we speak of an *indexed family*.

Let $a, r \in \hat{V}(S)$. If there exists one and only one $b \in \hat{V}(S)$ for which $\langle a, b\rangle \in r$, we write $r \uparrow a = b$; if this is not the case, we set $r \uparrow a = \varnothing$. The operation $\uparrow$ possesses the following properties:

1. If r is a function and $a \in \text{dom}(r)$, then $r \uparrow a = r(a)$.
2. $r \uparrow a \in \hat{V}(S)$ for all $r, a \in \hat{V}(S)$.

The *nonstandard extension* of the set of individuals S (denoted as $^*S$) can be built using the *ultrafilter construction*. We shall show the way it can be done by an example of the field of real numbers $\mathbb{R}$.

Let $\mathbb{R} \subseteq S$. Let I be some index set (here we put $I = \mathbb{N}$, but it can be in principle any infinite set), and $\mathscr{F}$ is a *free ultrafilter* on $\mathbb{N}$, i.e., $\mathscr{F} \in \mathscr{P}(\mathbb{N})$ is the set of subsets of $\mathbb{N}$ such that:

1. $\mathbb{N} \in \mathscr{F}$, $\varnothing \in \mathscr{F}$.
2. $A_1, \ldots, A_n \in \mathscr{F} \to A_1 \cap \cdots \cap A_n \in \mathscr{F}$.
3. $A \in \mathscr{F}$, $A \subseteq B \to B \in \mathscr{F}$.
4. $E = \{m_1, \ldots, m_n\}$, $m_i \in \mathbb{R} \to E \notin \mathscr{F}$.
5. $\forall E \subseteq \mathbb{N}$ or $E \in \mathscr{F}$ or $\mathbb{N} \backslash E \in \mathscr{F}$.

Now we define a new set $\mathbb{R}^{\mathbb{N}}$ as the set of all sequences $\{x_i\}_{i\in I}$ in $\mathbb{R}$, or, which is the same, as the set of all functions from I to $\mathbb{R}$. We shall say that two sequences f and g are equivalent ($f\sim g$) if

$$\{i\in\mathbb{N}|f_i=g_i\}\in\mathscr{F} \tag{2.2}$$

The equivalence relation $\sim$ defines classes of equivalence on $\mathbb{R}^{\mathbb{N}}$:

$$\bar{f}=\{g\in\mathbb{N}^{\mathbb{N}}|g\sim f\} \tag{2.3}$$

The quotient set $\mathbb{R}^{\mathbb{N}}$ by $\sim$ is denoted as $\mathbb{R}^{\mathbb{N}}/\mathscr{F}$ and called the *ultraproduct* of the set of individuals $\mathbb{R}$.

All operations defined on the full ordered field $\mathbb{R}$ take their interpretation in $\mathbb{R}^{\mathbb{N}}/\mathscr{F}$. For example,

$$f+g=c\leftrightarrow\{i\in\mathbb{N}|f_i+g_i=c_i\}\in\mathscr{F} \tag{2.4}$$

$$f\leq g\leftrightarrow\{i\in\mathbb{N}|f_i\leq g_i\}\in\mathscr{F} \tag{2.5}$$

etc.

The set $\mathbb{R}$ can be naturally put into $\mathbb{R}^{\mathbb{N}}/\mathscr{F}$ by the constant sequences

$$a\in\mathbb{R}\to\overline{\langle a,a,a,\ldots\rangle}\in\mathbb{R}^{\mathbb{N}}/\mathscr{F} \tag{2.6}$$

$\mathbb{R}^{\mathbb{N}}/\mathscr{F}$ is called then the 'nonstandard extension' of $\mathbb{R}$ and denoted as $^*\mathbb{R}$. We shall identify the usual real numbers $\mathbb{R}$ and their image in $^*\mathbb{R}$ (constant sequences). Then $\mathbb{R}\subseteq{}^*\mathbb{R}$.

$^*\mathbb{R}$ obviously is not isomorphic to $\mathbb{R}$. For example, an element $a=\langle 1,2,3,\ldots\rangle$ is such that $a>x$, $\forall x\in\mathbb{R}$. In other words, a is greater than any real number, and, for example, $b=\langle\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots\rangle$ is less than any $x\in\mathbb{R}$.

In general, the following subsets exist in $^*\mathbb{R}$:

1. Infinite numbers $\mathrm{Inf}(^*\mathbb{R})\subseteq{}^*\mathbb{R}$:

$$a\in\mathrm{Inf}(^*\mathbb{R})\to\forall n\in\mathbb{N},\quad |a|>n \tag{2.7}$$

2. Finite numbers $\mathrm{Fin}(^*\mathbb{R})\subseteq{}^*\mathbb{R}$:

$$a\in\mathrm{Fin}(^*\mathbb{R})\to\exists m\in\mathbb{N},\quad |a|<m \tag{2.8}$$

3. Infinitesimal numbers $\mathbf{I}\subseteq{}^*\mathbb{R}$:

$$a\in\mathbf{I}\to\forall n\in\mathbb{N}^+,\quad |a|<\frac{1}{n} \tag{2.9}$$

It is clear that $\mathbb{R}\subseteq\mathrm{Fin}(^*\mathbb{R})$, but $\mathrm{Fin}(^*\mathbb{R})$ contains also other, *near-standard* elements that do not lie in $\mathbb{R}$. It can be shown that if $x\in\mathrm{Fin}(^*\mathbb{R})$ and $x\notin\mathbb{R}$, there exists a single $y\in\mathbb{R}$ such that $|x-y|\in\mathbf{I}$. We shall write in such cases $x\approx y$.

As the superstructure $\hat{V}(S)$ was built under the individuals S, the superstructure $\hat{V}(*S)$ can be built in an analogous way by the substitution $S \to *S$ using the same definition.

We have seen that the properties of $\mathbb{R}$ can be translated into $*\mathbb{R}$. Also the specially formulated (by the means of the formal *language*) properties of the standard superstructure $\hat{V}(S)$ can be translated to $\hat{V}(*S)$. To proceed we shall formulate the notions of *language* and of the *Transfer Theorem.*

The language $L(\hat{V}(S))$ can be associated with the standard superstructure $\hat{V}(S)$ in which one can formulate assertions about $\hat{V}(S)$.

The *elementary formulas* in $L(\hat{V}(S))$ are the expressions

1. $a = b$
2. $\langle a, b \rangle$
3. $(a \uparrow b) = c$
4. $a \in b$

where a, b, c are *constants* (concrete elements) of $\hat{V}(S)$, or *variables* which can take any value from $\hat{V}(S)$.

Using the elementary formulas, one can generate the class of *all formulas* (or expressions) of $L(\hat{V}(S))$ by the help of the prepositions

& 'and'
$\to$ 'if'
or 'or'
$\leftrightarrow$ 'if and only if'
$\neg$ 'not'

and quantors

$\forall x$ 'for all x (in $\hat{V}(S)$)'
$\exists x$ 'there exists x (in $\hat{V}(S)$)'

by the rules

1. If Φ and Ψ are formulas in $L(\hat{V}(S))$, then $\Phi \& \Psi$, $\Phi \mathbf{or} \Psi$, $\neg\Phi$, $\Phi \to \Psi$, and $\Phi \leftrightarrow \Psi$ are also formulas in $L(\hat{V}(S))$.
2. If Φ is a formula in $L(\hat{V}(S))$ and x is a variable, then $\exists x \Phi$ and $\forall x \Phi$ also are formulas in $L(\hat{V}(S))$.

The most general form of a formula of the language $L(\hat{V}(S))$ is

$$\Phi = \Phi(x_i, \ldots, x_n) \tag{2.10}$$

where x_i are *free variables* (i.e., not restricted by the quantors **&** and **or**) that take their values in $\hat{V}(S)$.

Every formula has the direct interpretation in $\hat{V}(S)$:

if $\Phi(x)$ is a formula in $L(\hat{V}(S))$ and $A \in \hat{V}(S)$, then the expression $\Phi(A)$ expresses an assertion about A which is true in $\hat{V}(S)$

For example, we shall say that 'X is a set' if

$$\Phi(X) = \{(X = 0) \textbf{ or } (\exists x \in X)(x = x)\} \tag{2.11}$$

So, if $A \in \hat{V}(S)$ and $\Phi(A)$ is true, then A is a set in $\hat{V}(S)$. On the other hand, if A is a set in $\hat{V}(S)$, then $\Phi(\mathrm{A})$ is true.

In such a way all the necessary expressions and propositions about $\hat{V}(S)$ can be generated.

But formulas of the language $L(\hat{V}(S))$ can be also interpreted in the extended superstructure $\hat{V}(^*S)$!

Indeed, one can proceed by the help of a construction which is analogous to the ultrafilter construction for individuals. Now it is called the *limited ultrafilter construction.*

A sequence of the elements of $\hat{V}(S)$ $\{A(i)|i \in I\}$ is *limited* if there exists a fixed number $m \in \mathbb{N}$ such that $(\forall i \in I)(A(i) \in V_m(S))$. The set of all limited sequences is denoted as $\hat{V}(S)^I$. Two limited sequences are equivalent according to the free ultrafilter $\mathscr{F}$ on I, $A \sim_{\mathscr{F}} B$, if $\{i \in I|A(i) = B(i)\} \in \mathscr{F}$.

We shall denote classes of equivalence of the sequences of $\hat{V}(S)^I$ as $A_{\mathscr{F}} = \{B \in \hat{V}(S)|B \sim_{\mathscr{F}} A\}$. Then by the *limited ultraproduct* $\hat{V}(S)^I/\mathscr{F}$ we shall call the quotient set $\hat{V}(S)^I$ by $\sim_I$.

In the limited ultraproduct $\hat{V}(S)^I/\mathscr{F}$ the elementary formulas of $L(\hat{V}(S))$ can be interpreted:

$$\begin{gathered}
A_{\mathscr{F}} = B_{\mathscr{F}} \leftrightarrow \{i \in I|A(i) = B(i)\} \in \mathscr{F} \\
\langle A_{\mathscr{F}}, B_{\mathscr{F}} \rangle = C_{\mathscr{F}} \leftrightarrow \{i \in I|\langle A(i), B(i)\rangle = C(i)\} \in \mathscr{F} \\
(A_{\mathscr{F}} \uparrow B_{\mathscr{F}}) = C_{\mathscr{F}} \leftrightarrow \{i \in I|(A(i) \uparrow B(i)) = C(i)\} \in \mathscr{F} \\
A_{\mathscr{F}} \in B_{\mathscr{F}} \leftrightarrow \{i \in I|A(i) \in B(i)\} \in \mathscr{F}
\end{gathered} \tag{2.12}$$

The prepositions and quantors conserve their logical meaning under all sets; then one can interpret in $\hat{V}(S)^I/\mathscr{F}$ the class of *all formulas* of $L(\hat{V}(S))$.

There exists a natural embedding of the standard superstructure into the limited ultrapower:

$$i\colon \quad \hat{V}(S) \mapsto \hat{V}(S)^I/\mathscr{F} \tag{2.13}$$

where $i(A) = \langle A, A, A, \ldots\rangle_{\mathscr{F}}$ is a class of equivalence of the constant sequence of elements of $\hat{V}(S)$.

However, the limited ultrapower $\hat{V}(S)^I/\mathscr{F}$ is not isomorphic to the nonstandard superstructure $\hat{V}(^*S)$. There exist sets in $\hat{V}(^*S)$ (the so-called *external sets*) that have no domain in $\hat{V}(S)^I/\mathscr{F}$. But there exists an embedding $\hat{V}(S)^I/\mathscr{F}$ into $\hat{V}(*S)$

$$j: \quad \hat{V}(S)^I/\mathscr{F} \mapsto \hat{V}(*S) \tag{2.14}$$

such that

1. If $A_{\mathscr{F}} \in *S$, then $j(A_{\mathscr{F}}) = A_{\mathscr{F}}$, where $*S = S^I/\mathscr{F} \in \hat{V}(S)^I/\mathscr{F}$
2. If $A_{\mathscr{F}} \notin *S$, then $j(A_{\mathscr{F}}) = \{j(B_{\mathscr{F}}) | B_{\mathscr{F}} \in A_{\mathscr{F}}\}$.

The mapping j is defined by induction; it conserves all the elementary formulas that are interpreted in the limited ultraproduct $\hat{V}(S)^I/\mathscr{F}$.

Combining the mappings i and j, one receives a new 'star' *-mapping—$*A = j(i(A))$ from $\hat{V}(S)$ into $\hat{V}(*S)$. The *-mapping satisfies the *Transfer Theorem.*

Let $\Phi(X_1, X_2, \ldots, X_k)$ be a formula in $L(\hat{V}(S))$, where X_i are free variables. Giving to every X_i either the value A_i from $\hat{V}(S)$ or $*A_i = j(i(A_i))$ from $\hat{V}(*S)$, one interprets this formula either in $\hat{V}(S)$ or in $\hat{V}(*S)$.

Theorem 1 (Transfer Theorem). Let $A_1, \ldots, A_n \in \hat{V}(S)$. Then any assertion Φ in $L(\hat{V}(S))$ which is true for $A_1, \ldots, A_n$ in $\hat{V}(S)$ is also true for $*A_1, \ldots, *A_n$ in $\hat{V}(*S)$ and vice versa.

Sets of the nonstandard superstructure $\hat{V}(*S)$ are divided into three classes according to the star *-mapping:

1. A set A is called *standard* if $\exists B \in \hat{V}(S)$ $(A = *B)$.
   Standard sets in $\hat{V}(*S)$ are those that can be obtained from the constant sequences of the elements of $\hat{V}(S)$.
2. A set A is called *internal* if $\exists B \in \hat{V}(S)$ $(A \in *B)$.
   Internal sets can be obtained from the limited sequences of elements of $\hat{V}(S)$.
3. A set A is *external* if it is not an internal one.

So assertions about sets in $\hat{V}(S)$ can be transferred to the assertions about the internal sets in $\hat{V}(S)$, and vice versa.

One can say *a priori* nothing about the external sets (if they exist) in $\hat{V}(*S)$. The answer to the question of the existence of external sets [and in general, of the existence of nonstandard elements in $\hat{V}(S)$] depends on the choice of the index set I and the ultrafilter on it. We shall not consider that here. We shall intend, and it is enough for our purposes, that I and the ultrafilter on it are such that they allow the existence of nonstandard elements by extending the real axis $\mathbb{R}$. It is known that such a choice is possible (Davis, 1977; Albeverio *et al.*, 1950).

We list now briefly the properties of the internal sets in $\hat{V}(*S)$ and of the *-mapping.

By definition, all the internal sets are contained in the image of the standard superstructure $*\hat{V}(S)$ in the extended superstructure $\hat{V}(S)$. The subset of the extended superstructure $*\hat{V}(S)$ we shall call the *nonstandard universum*, and its domain $\hat{V}(S)$ the *standard universum*.

We shall call the set A *definable* in $\hat{V}(S)$ if there exists a formula $\Phi(X)$ in $L(\hat{V}(S))$ such that

$$A = \{b \in \hat{V}(S) | \Phi(b)\} \tag{2.15}$$

It can be shown that

$$*A \equiv \{b \in *\hat{V}(S) | \Phi(b)\} \tag{2.16}$$

and the mapping * defined in such a way does not depend on the concrete formula Φ (Davis, 1977). Exactly so we shall build concrete internal sets in $\hat{V}(*S)$, which are called *extensions* of sets from $\hat{V}(S)$.

The word 'extension' is justified by the following properties of the * operation:

1. If $A \subseteq S$, then $A \subseteq *A$ and $*A \cap S = A$.
2. The mapping * conserves the $\cup$, $\cap$, $\subseteq$, and $\setminus$ operations.
3. Functions[2] $f \in \hat{V}(S)$, which are defined on individuals S and have their range also in S, can be continued in $\hat{V}(*S)$ from S^m to $*S^m$ (Davis, 1977).

That is, if $x, y \in S$ and if $*f(*x) = *y$, and because $*x = x$, $*y = y$, one has $*f(x) = y$. Thus, we shall omit a star at such an extended function $*f \in \hat{V}(*S)$. For example, we omit stars at the operation $+$, $\cdot$ on the field $*\mathbb{R}$.

The set of individuals may contain subsets that are of a different nature. For example, $S = \mathbb{H} \cup \mathbb{C}$, where $\mathbb{H}$ is a Hilbert space and $\mathbb{C}$ the field of complex numbers. Then we define in an analogous way the extended superstructure $\hat{V}(*\mathbb{H} \cup *\mathbb{C})$ and the mapping *.

Conclusion 1. If there is some mathematical structure that is defined by axioms (i.e., by formulas) in the standard universum, then the extension of this structure is a subset of the nonstandard universum, which is defined by the same set of axioms, but interpreted yet in the nonstandard universum.

The scheme of doing this can be as follows:

1. Choose the set of individuals S and build the standard superstructure $\hat{V}(S)$.

[2]The notion of a *function* is in fact an abbreviation of its full definition in the formal language $L(\hat{V}(S))$. One can find it, for example, in Davis (1977).

2. Build the extended set of individuals $*S$ and the nonstandard superstructure $\hat{V}(*S)$.
3. Define the necessary mathematical structure in the standard superstructure by the set of axioms (formulas) $\{\Phi_i\}_{i=1}^n$, formulated in the formal language $L(\hat{V}(S))$.
4. Translate this axiom system to the nonstandard universum using the Transfer Theorem to receive a new extended structure, which is defined by these axioms.

We shall use this scheme for the extension of quantum logics in the next sections.

3. THE FORMALIZED QUANTUM LOGIC SCHEME

In this section we shall formulate all the axioms and conclusions of the quantum logic (Beltrametti and Casinelli, 1976) in the language $L(\hat{V}(S))$ that lead to the separate Hilbert space formalism of quantum mechanics. Its main result is formulated in the language $L(\hat{V}(S))$ algebraic structure of quantum mechanics, which we shall call the $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \hat{\mathscr{G}} \rangle$ theory. From the point of view of the nonstandard analysis, it is a set of formulas $\{\Phi_i\}_{i=1}^n$ about the elements of superstructure $\hat{V}(S)$. As we said in the previous section, using the Transfer Theorem, we shall be able to build an extended (nonstandard) quantum logic afterward in the nonstandard universum $*V(S)$.

3.1. Preliminary Considerations—$\langle \mathscr{O}, \hat{\mathscr{G}}, p(\cdot) \rangle$ Theory

The following assumptions form the basis of the axiomatic approach to quantum mechanics.

1. The following abstract sets exist, which define every physical system: $\mathscr{O}$, the set of observables; $\hat{\mathscr{G}}$, the set of states; $\mathscr{B}(\mathbb{R})$, the set of intervals of the real axis $\mathbb{R}$ where the observables take their values in after measurement.

Axiom 1. $\mathscr{B}(\mathbb{R})$ is a family of the Borel subsets of $\mathbb{R}$, i.e.:

1. Union and intersection of not more than a countable number of elements of $\mathscr{B}(\mathbb{R})$ are contained in $\mathscr{B}(\mathbb{R})$.
2. $\mathbb{R}\backslash E \in \mathscr{B}(\mathbb{R})$ for any $E\mathscr{B}(\mathbb{R})$.
3. Any open ball $B_r(x) = \{y \in \mathbb{R} \mid |x-y| < r\}$ belongs to $\mathscr{B}(\mathbb{R})$.

2. There exists a function $p(A, \alpha, E)$: $\mathscr{O} \times \hat{\mathscr{G}} \times \mathscr{B}(\mathbb{R}) \mapsto [0, 1] \subseteq \mathbb{R}$, which is a probability that in the process of measuring the observable A of our physical system in the state $\hat{\alpha}$, its values lie within the interval E from $\mathscr{B}(\mathbb{R})$.

There exists a set $\mathscr{E} = \mathcal{O} \times \mathscr{B}(\mathbb{R})$, $a \in \mathscr{E} \leftrightarrow a = \langle A, E \rangle$, which is called the set of *propositions*. Then $\langle A, E \rangle$ is the question: Does the value of the observable A lie in the interval E? On $\mathscr{E}$ a function $p_{\hat{\alpha}}(\cdot)$ is defined: $p_{\hat{\alpha}}(a) = p(A, \hat{\alpha}, E)$, $a = \langle A, E \rangle$. We shall take $x \sim y$ for $x \in \mathscr{E}, y \in \mathscr{E}$, if

$$\forall \hat{\alpha} \in \mathscr{G} \quad p_{\hat{\alpha}}(x) = p_{\hat{\alpha}}(y) \tag{3.1}$$

Let $\mathscr{L} = \mathscr{E}/\sim$ be the quotient set $\mathscr{E}$ by the equivalence relation $\sim$. The set $\mathscr{L}$ is called the *logic* of the quantum system. We shall denote by $\bar{x} \in \mathscr{L}$ the class of propositions as

$$\bar{x} = \{a \in \mathscr{E} \,|\, a \sim x\}, \qquad x \in \mathscr{E} \tag{3.2}$$

Also the operation of orthocomplementation can be defined on $\mathscr{E}$:

$$a = \overline{\langle A, E \rangle} \rightarrow a^{\perp} = \langle A, \mathbb{R} \backslash E \rangle \tag{3.3}$$

The element $a^{\perp}$ stands for the proposition which is the negation of the initial proposition a.

Axiom 2. $\forall A \in \mathcal{O}, \forall \hat{\alpha} \in \hat{\mathscr{G}}, p(A, \hat{\alpha}, E)$ is a probability measure on $\mathscr{B}(\mathbb{R})$:

1. $p(A, \hat{\alpha}, \varnothing) = 0$, $p(A, \hat{\alpha}, \mathbb{R}) = 1$.
2. If $\{E_i \,|\, i \in \mathbb{N}\} \subseteq \mathscr{B}(\mathbb{R})$ and $E_i \cap E_j = \varnothing$, $i \neq j$, then

$$p\left(A, \hat{\alpha}, \bigcup_{i \in \mathbb{N}} E_i\right) = \sum_{i \in \mathbb{N}} p(, \hat{\alpha}, E_i) \tag{3.4}$$

Axiom 3. If for all observables $A \in \mathcal{O}$ and for all the intervals from $\mathscr{B}(\mathbb{R})$ the probability $p(A, \hat{\alpha}, E) = p(A, \hat{\beta}, E)$ is the same for two states $\hat{\alpha}$ and $\hat{\beta}$, then these two states coincide.

Let us build a set of functions $\mathscr{F}$ from $\mathscr{L}$ to $[0, 1]$ as

$$\mathscr{G} = \{\hat{\alpha}: \mathscr{L} \mapsto [0, 1] \,|\, \hat{\alpha}(a) = p_{\hat{\alpha}}(a) \; \forall a \in \mathscr{L}\} \tag{3.5}$$

and the set of mappings from $\mathscr{B}(\mathbb{R})$ to $\mathscr{L}$:

$$\mathscr{X} = \{\mu_A : \mathscr{B}(\mathbb{R}) \mapsto \mathscr{L} \,|\, \mu_A(E) = \overline{\langle A, E \rangle}, \forall A \in \mathcal{O}, \forall E \in \mathscr{B}(\mathbb{R})\} \tag{3.6}$$

The set $\mathscr{G}$ is called the set of *probability measures* on the logic $\mathscr{L}$, and $\mathscr{X}$ is the set of *$\mathscr{L}$-valued measures* on $\mathscr{B}(\mathbb{R})$. From now on we shall identify these sets with the set of states $\hat{\mathscr{G}}$ and with the set of observables $\mathcal{O}$, respectively.

Axiom 4. For every sequence of elements of $\mathscr{L}$, $\{a_i \,|\, i \in \mathbb{N}\} \subseteq \mathscr{L}$, such that for every state $\hat{\alpha}$ from $\hat{\mathscr{G}}$,[3]

$$\hat{\alpha}(a_i) + \hat{\alpha}(a_j) \leq 1, \qquad i \neq j^2 \tag{3.7}$$

[3]We say that a is orthogonal to b ($a \perp b$) if $\hat{\alpha}(a) + \hat{\alpha}(b) \leq 1$, $\forall \hat{\alpha}(\cdot) \in \mathscr{G}$.

there exists $a \in \mathscr{L}$ such that

$$\hat{\alpha}(a) = \sum_{i \in \mathbb{N}} \hat{\alpha}(a_i) \tag{3.8}$$

Axiom 5. Every σ-convex combination of the states of $\mathscr{G}$ is also contained in $\mathscr{G}$, in other words,

$$\forall\{\hat{\alpha}_i(\,\cdot\,)|i \in \mathbb{N}\} \subseteq \mathscr{G}, \quad \forall\{t_i|i \in \mathbb{N}\} \subseteq \mathbb{R}, \quad \text{such that} \quad \sum_{i \in \mathbb{N}} t_i = 1 \tag{3.9}$$

there exists $\hat{\alpha}(\,\cdot\,) \in \mathscr{G}$ such that

$$\hat{\alpha}(a) = \sum_{i \in \mathbb{N}} t_i \hat{\alpha}_i(a) \qquad \text{for all} \quad a \in \mathscr{L} \tag{3.10}$$

These five axioms define the mathematical structure that contains the logic $\mathscr{L}$, the probability measures $\mathscr{G}$ on $\mathscr{L}$, and $\mathscr{L}$-valued measures $\mathscr{X}$ on $\mathscr{B}(\mathbb{R})$.

Conclusion 2. The logic $\mathscr{L}$ has the following properties:

1. $\mathscr{L}$ is a partially ordered set.
2. $\mathscr{L}$ contains the least element $\hat{\mathbf{0}}$ and the greatest element $\hat{\mathbf{1}}$ such that $\hat{\mathbf{0}} \leq a < \hat{\mathbf{1}}$ for any $a \in \mathscr{L}$.
3. The set $\mathscr{L}$ possesses an orthocomplementation map $^\perp$.
4. The least upper bound (see definition in Section 3.2) $a = \bigvee_{i \in \mathbb{N}} a_i$ exists in $\mathscr{L}$ for any countable set of its mutually orthogonal elements ($a_i \perp a_j$).
5. $\mathscr{L}$ is an orthomodular set (Beltrametti and Casinelli, 1976).

Conclusion 3. The properties of the order-defining σ-convex family of probability measures $\mathscr{G}$ are:

1. $(\forall a, b \in \mathscr{L})\ (\forall \hat{\alpha}(\,\cdot\,) \in \mathscr{G})\ (a \leq b \leftrightarrow \forall \hat{\alpha}(\,\cdot\,) \in \mathscr{G}\ \hat{\alpha}(a) < \hat{\alpha}(b))$.
2. $(\forall\{\hat{\alpha}_i(\,\cdot\,)|i \in \mathbb{N}\} \subseteq \mathscr{G})\ (\forall(t_i|i \in \mathbb{N}\} \subseteq \mathbb{R}^+)\ (\sum_{i \in \mathbb{N}} t_i = 1)$

$$(\exists \hat{\alpha}(\,\cdot\,) \in \mathscr{G}) \quad \left(\hat{\alpha} = \sum_{i \in \mathbb{N}} t_i \hat{\alpha}_i\right) \tag{3.11}$$

3. $\hat{\alpha}(\hat{\mathbf{0}}) = 0$, $\hat{\alpha}(\hat{\mathbf{1}}) = 1$, for any $\hat{\alpha}(\,\cdot\,)$ from $\mathscr{G}$.
4. $(\forall \hat{\alpha}(\,\cdot\,) \in \mathscr{G})\ (\forall\{a_i\}_{i \in \mathbb{N}} \subseteq \mathscr{L})\ (i \neq j \rightarrow a_i \perp a_j)$

$$\hat{\alpha}\left(\bigvee_{i \in \mathbb{N}} a_i\right) = \sum_{i \in \mathbb{N}} \hat{\alpha}(a_i) \tag{3.12}$$

A state $\hat{\alpha} \in \mathscr{G}$ is called a *pure* one if it is not a convex combination of other states from $\mathscr{G}$. Otherwise it is called *mixed*.

Conclusion 4. The properties of the set of $\mathscr{L}$-valued measures $\mathscr{X}$ are:

1. $(\forall E_1 \in \mathscr{B}(\mathbb{R}))\ (\forall E_2 \in \mathscr{B}(\mathbb{R}))\ (\forall \mu \in \mathscr{X})$

$$E_1 \cap E_2 = \varnothing \to \mu(E_1) \perp \mu(E_2) \tag{3.13}$$

2. $\forall \mu \in \mathscr{X},\ \forall \{E_i | i \in \mathbb{N}\} \subseteq \mathscr{B}(\mathbb{R})\ (i \neq j \to E_i \cap E_j = \varnothing)$

$$\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) = \bigvee_{i \in \mathbb{N}} \mu(E_i) \tag{3.14}$$

3. $(\forall \mu \in \mathscr{X})$

$$(\mu(\varnothing) = \hat{\mathbf{0}})\ \&\ (\mu(\mathbb{R}) = \hat{\mathbf{1}}) \tag{3.15}$$

3.2. Axioms of the $\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle$ Theory

In this section we continue to formulate the axioms concerning the structure of the logic $\mathscr{L}$ needed for the formulation of the quantum logic of the separable Hilbert space quantum mechanics.

Let the binary operations exist on $\mathscr{L}$ (the least upper bound and the greatest lower bound)

$$\bigvee \uparrow \langle a, b \rangle \equiv a \vee b \in \mathscr{L} \tag{3.16}$$

$$\bigwedge \uparrow \langle a, b \rangle \equiv a \wedge b \in \mathscr{L} \tag{3.17}$$

$\mathrm{dom}(\vee) = \mathrm{dom}(\wedge) = \mathscr{L} \times \mathscr{L}$, $\mathrm{ran}(\vee) = \mathrm{ran}(\wedge) = \mathscr{L}$. Their properties are defined in the following axiom.

Axiom 6. The logic of the physical system $\mathscr{L}$ possesses the $\vee$ and $\wedge$ operations with the following properties:

1. *Idempotention*: $(\forall x \in \mathscr{L})$

$$(x \wedge x = x)\ \&\ (x \vee x = x) \tag{3.18}$$

2. *Commutativity*: $(\forall x \in \mathscr{L})\ (\forall y \in \mathscr{L})$

$$(x \wedge y = y \wedge x)\ \&\ (x \vee y = y \vee x) \tag{3.19}$$

3. *Associativity*: $(\forall x \in \mathscr{L})\ (\forall y = \mathscr{L})\ (\forall z \in \mathscr{L})$

$$\{x \wedge (y \wedge z) = (x \wedge y) \wedge z\}\ \&\ \{x \vee (y \vee z) = (x \vee y) \vee z\} \tag{3.20}$$

4. *Completeness*: $(\forall A \in \mathscr{P}(\mathscr{L}))\ (\exists a \in \mathscr{L})\ (\forall x \in A)$

$$x \wedge a = a \tag{3.21}$$

$\&\ (\exists t \in \mathscr{L})\ (\forall x \in A)$

$$x \wedge t = t \to a \wedge t = t \tag{3.22}$$

and dually for $\vee$.

The next axiom determines the compatibility of the existing partial ordering on $\mathscr{L}$ and a new one defined by the new operations $\vee$ and $\wedge$.

Axiom 7. Compatibility of the partial ordering definitions on $\mathscr{L}$: $(\forall x \in \mathscr{L})\ (\forall y \in \mathscr{L})$

$$x \leq y \leftrightarrow x \wedge y = x \leftrightarrow x \vee y = y \tag{3.23}$$

The properties of the previously defined operation of orthocomplementation are specified in the next axiom:

Axiom 8. The orthocomplementation on $\mathscr{L}$ has the following properties:

1. $(\forall x \in \mathscr{L})$

$$(x^{\perp})^{\perp} = x \tag{3.24}$$

2. $(\forall x \in \mathscr{L})\ (\forall y \in \mathscr{L})$

$$\{(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}\}\ \&\ \{(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}\} \tag{3.25}$$

3. $(\forall x \in \mathscr{L})$

$$(x \vee x^{\perp} = \hat{\mathbf{1}})\ \&\ (x \wedge x^{\perp} = \hat{\mathbf{0}}) \tag{3.26}$$

The introduced axioms imply that the logic of the physical system $\mathscr{L}$ is a *lattice*. The last axiom in this subsection specifies the properties of this lattice.

Axiom 9. 1. The lattice $\mathscr{L}$ is orthomodular: $(\forall a \in \mathscr{L})\ (\forall b \in \mathscr{L})$

$$a \leq b \rightarrow b = a \vee (b \wedge a^{\perp}) \tag{3.27}$$

2. The lattice $\mathscr{L}$ is atomic: $(\forall x \in \mathscr{L})\ (\exists p \in \mathscr{L})$,

$$(p \text{ covers } \hat{\mathbf{0}})\ \&\ (p \leq x) \tag{3.29}$$

3. The lattice $\mathscr{L}$ possesses the covering property[4]: $(\forall x \in \mathscr{L})\ (\forall p \in \mathscr{L})$

$$\{(p \text{ covers } \hat{\mathbf{0}})\ \&\ (\neg(p \leq x)\} \rightarrow (x \vee p \text{ covers } x)) \tag{3.30}$$

4. The lattice $\mathscr{L}$ is irreducible: $\forall a \in \mathscr{L}$

$$\{(\exists z \in \mathscr{L})\ (z = (z \wedge a) \vee (z \wedge a^{\perp}))\} \rightarrow (z = \hat{\mathbf{0}}) \textbf{ or } (z = \hat{\mathbf{1}}) \tag{3.31}$$

Thus, the cited axioms say that the logic $\mathscr{L}$ is a complete irreducible orthomodular lattice with the covering property (a complete irreducible

[4] "b covers a in $\mathscr{L}$" means that $(\forall a \in \mathscr{L})\ (\forall b \in \mathscr{L})$

$$(\exists c \in \mathscr{L})\ (a \leq c)\ \&\ (c \leq b) \rightarrow (c = b) \textbf{ or } (c = a) \tag{3.28}$$

OAC-lattice). Remember that the structure $\langle \mathscr{L}, \leq, {}^{\perp}, \vee, \wedge \rangle$ is defined now in the language $L(\hat{V}(S))$.

3.3. The $\mathscr{L}(\mathbb{H})$ Lattice

There exists a connection between the lattice $\mathscr{L}$ and the lattices of the subspaces of vector spaces.

Let $\mathbb{H}$ be a vector space under the division ring K and with the involutive antiautomorphism $\lambda \to \tilde{\lambda}, \lambda \in K$ (Maçzynski, 1972; Beltrametti and Casinelli, 1976). The Hermitian form $f(\cdot, \cdot)$ is defined on $\mathbb{H}$ with its properties (Akhiezer and Glazman, 1950). Its formalized definition in the language $L(\hat{V}(S))$ can be found in Davis (1977), for instance. We suppose that $\mathbb{H} \subseteq S, K \subseteq S$.

Introduce a unary operation on $\mathscr{P}(\mathbb{H})$:

$$^\circ \uparrow a = a^\circ \in \mathscr{P}(\mathbb{H}), \qquad \mathrm{dom}(^\circ) = \mathscr{P}(\mathbb{H}) \tag{3.32}$$

such that $(\forall a \in \mathscr{P}(\mathbb{H}))$

$$a^\circ = \{b \in \mathbb{H} | f(c, b) = 0, \forall c \in a\} \tag{3.33}$$

The subspace u of $\mathbb{H}$ is called *closed* if $u^{\circ\circ} = u$. Define the set of all closed subspaces in $\mathbb{H}$ as

$$\mathscr{L}(\mathbb{H}) = \{u \in \mathscr{P}(\mathbb{H}) | u^{\circ\circ} = u\} \tag{3.34}$$

Binary operations $\vee$ and $\wedge$ are defined on $\mathscr{L}(\mathbb{H})$ such that for $\forall a \in \mathscr{L}(\mathbb{H})$, $\forall b \in \mathscr{L}(\mathbb{H})$,

$$(a \vee b = (a + b)^{\circ\circ}) \;\&\; (a \wedge b \equiv a \cap b) \tag{3.35}$$

Here $a + b$ is a linear envelope of the subspaces a and b.

Also the partial ordering is defined on $\mathscr{L}(\mathbb{H})$ by inclusion: for $\forall a \in \mathscr{L}(\mathbb{H})$, $\forall b \in \mathscr{L}(\mathbb{H})$,

$$a \leq b \leftrightarrow a \subseteq b \tag{3.36}$$

Thus, we see that $\mathscr{L}(\mathbb{H})$ is a lattice. Moreover, the following theorem holds (see, for instance, Piron, 1976; Beltrametti and Casinelli, 1976).

Theorem 2. Let $\mathscr{L}$ be a complete irreducible OAC-lattice with length greater than 4. Then there exist a division ring K with involutive antiautomorphism $\lambda \to \tilde{\lambda}$ $(\lambda \in K)$ and a vector space $\mathbb{H}$ on K with the well-defined Hermitian form $f(\cdot, \cdot)$ such that $\mathscr{L}$ is orthoisomorphic to the lattice $\mathscr{L}(\mathbb{H})$ of closed subspaces of $\mathbb{H}$.

Axiom 10. The ring K defined by the lattice $\mathscr{L}$ is the field of complex numbers $\mathbb{C}$.

Then, using the Amemiya and Araki (1986) theorem, we conclude that $\mathbb{H}$ is nothing else but a Hilbert space, and the defined Hermitian form f is the scalar product $(\cdot, \cdot)$ in $\mathbb{H}$.

Formalizing this in the language $L(\hat{V}(S))$, we say:

Conclusion 5. 1. $\mathscr{L} \subseteq S$, $\mathbb{H} \subseteq S$, $\mathbb{C} \subseteq S$, where $\mathscr{L}$ is a subset of the set of individuals S, which is defined by Axioms 6–9, and $\mathbb{H}$ is a complete Hilbert space on the field $\mathbb{C}$ [for a full axiomatic description of $\mathbb{H}$ and $\mathbb{C}$, see Davis (1979)].

2. There exists a bijection

$$\mu: \quad \mathscr{L} \overset{1-1}{\longmapsto} \mathscr{L}(\mathbb{H}) \tag{3.37}$$

such that

(a) $\forall a \in \mathscr{L}, \forall b \in \mathscr{L}\ (a \leq b \leftrightarrow \mu(a) \leq \mu(b))$.
(b) $\forall a \in \mathscr{L}\ (\mu(a^{\perp}) = (\mu(a))^{\circ})$.
(c) $\forall a \in \mathscr{L}, \forall b \in \mathscr{L}\ (\mu(a \vee b) = \mu(a) \vee \mu(b))\ \&\ (\mu(a \wedge b) = \mu(a) \wedge \mu(b))$.

3.4. Observables and States in $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \mathscr{G} \rangle$ Theory

The states can be characterized by the help of the Gleason (1957) theorem and the observables by the projector-valued measures.

It is known that to every closed subspace u in $\mathbb{H}$ an orthogonal projector can be brought to coincidence (Akhiezer and Glazman, 1950). Thus, to every $a \in \mathscr{L}(\mathbb{H})$ there exists a projector P^a: $\mathbb{H} \mapsto \mathbb{H}$. One can put to coincidence with every $\mathscr{L}$-valued measure x a projector-valued measure

$$P_E \equiv P^{x(E)}: \quad \mathscr{B}(\mathbb{R}) \mapsto \{P^M\}_{M \in \mathscr{L}(\mathbb{H})} \tag{3.38}$$

with the following properties:

1.

$$P_{\varnothing} = 0, \qquad P_{\mathbb{R}} = 1 \tag{3.39}$$

2. $\forall E_1 \in \mathscr{B}(\mathbb{R}), \forall E_2 \in \mathscr{B}(\mathbb{R})$,

$$E_1 \cap E_2 = \varnothing \rightarrow P_{E_1} P_{E_2} = P_{E_2} P_{E_1} = 0 \tag{3.40}$$

3. $\forall \{E_i \in \mathscr{B}(\mathbb{R}) | i \in \mathbb{N}\}\ (i \neq j \rightarrow E_i \cap E_j = \varnothing)$

$$P_{\bigcup E_i} = \sum_{i \in \mathbb{N}} P_{E_i} \tag{3.41}$$

By their definition and using the isomorphism $\mathscr{L} \sim \mathscr{L}(\mathbb{H}) \sim \{P^M\}$, the states are the probability measures on the projectors. By the Gleason theorem, if $\dim(\mathbb{H})$ is greater than 3, the set of pure states is exactly the set of functions

$$m_\phi:\quad \mathscr{L}(\mathbb{H}) \mapsto [0, 1], \qquad \text{where } \phi \in \mathbb{H}, \quad \|\phi\| = 1 \tag{3.42}$$

such that

$$m_\phi(M) = (\phi, P^M\phi), \qquad M \in \mathscr{L}(\mathbb{H}) \tag{3.43}$$

The set of all states is therefore the set of all convex combinations of the pure states m_ϕ.

To this end, the previous axioms were defined only for not more than countable sets. Thus we are allowed to speak only about the case of the separable Hilbert space $\mathbb{H}$. But for the necessity of our axiom system we need nevertheless an axiom of separability.

Axiom 11. The Hilbert space $\mathbb{H}$ determined by the previous axiom system is separable.

4. SPECTRAL PROPERTIES OF OBSERVABLES IN $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \mathscr{G}$ THEORY

In this section, spectral properties of observables ($\mathscr{L}$-valued measures) are considered. By the *spectrum* $s(x)$ of the observable x we mean, roughly speaking, the minimal subset of the set of real numbers $\mathbb{R}$ such that $x(s(x)) = \hat{\mathbf{1}}$. It is clear that if $I \nsubseteq s(x)$, then $x(I) = \hat{\mathbf{0}}$.

In the separate Hilbert space formalism, if $s(x)$ is a countable subset of $\mathbb{R}$, then the so-called *eigenstate* $\lambda_i \in \mathscr{G}$ corresponds to every $\lambda_i \in s(x)$ with its definite properties (von Neumann, 1955). If in the next turn $s(x)$ is uncountable, the definition of the family of eigenstates parametrized by points from $s(x)$ cannot be carried out properly.

On the contrary, the Dirac formalism of quantum mechanics allows us to define such families $\{\alpha_\lambda | \lambda \in s(x)\}$ for observables either with discrete or continuous spectra.

We shall demonstrate here that the quantum logic $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \mathscr{G} \rangle$ formulated above does not allow the definition of the eigenstates for the observables with continuous spectra. It means, therefore, that Dirac's formalism cannot be derived from this version of quantum logic.

Some more definitions follow.

The *resolvent set* for an observable $x \in \mathscr{X}$ is the set

$$r(x) = \bigcup \{I \in \mathscr{B}(\mathbb{R}) | \alpha(x(I)) = 0, \forall \alpha \in \mathscr{G}\} \tag{4.1}$$

The *spectrum* of an observable $x \in \mathscr{X}$ is the set

$$s(x) = \mathbb{R} \backslash r(x) \tag{4.2}$$

The spectrum $s(x)$ is *discrete* if $s(x) \subseteq \mathbb{R}$ is not more than a countable subset of $\mathbb{R}$. It is *continuous* if it is not discrete. A continuous spectrum is called *mixed* if it contains isolated points. A continuous spectrum is *purely continuous* if it is not mixed.

The spectrum of any observable can be decomposed into discrete and purely continuous parts s_p and s_c

$$s(x) = s_p(x) \cup s_c(x) \tag{4.3}$$

4.1. Observables with Discrete Spectra

Lemma 1. Let λ be an isolated point in the spectrum of an observable x. Then $\exists \epsilon \in \mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+$

$$\delta \leq \epsilon \rightarrow x((\lambda - \delta, \lambda + \delta)) = x((\lambda - \epsilon, \lambda + \epsilon)) \tag{4.4}$$

Moreover, if $\exists \lambda \in \mathbb{R}$, $\exists \epsilon \in \mathbb{R}^+$ such that for $\forall \delta \in \mathbb{R}^+$

$$\delta \leq \epsilon \rightarrow x((\lambda - \delta, \lambda + \delta)) = x((\lambda - \epsilon, \lambda + \epsilon)) \tag{4.5}$$

then λ is an isolated point in the spectrum of $x \in \mathscr{X}$.

Proof. Let $E \in \mathscr{B}(\mathbb{R})$, $\lambda \in E$, and there is no other value of $s(x)$ lying in E. Such an interval E always can be chosen for λ isolated. Let $E' \subset E$ and $E' \backslash E = \Delta$.

Then $E' \cup \Delta = E$, $E \cap \Delta = \varnothing$. Note that either $\lambda \notin \Delta$, $\lambda \in E'$ or $\lambda \in \Delta$, $\lambda \notin E'$.

By the properties of $\mathscr{L}$-valued measures (Conclusion 4)

$$x(E) = x(E' \cup \Delta) = x(E') \vee x(\Delta) \tag{4.6}$$

Because either Δ or E' belongs to the resolvent $r(x)$, and for definiteness choosing $\lambda \in E'$, we have

$$x(\Delta) = \hat{\mathbf{0}} \qquad \text{and} \qquad x(E) = x(E') \qquad \forall E' \subset E \tag{4.7}$$

If $\forall \delta < \epsilon$, $x((\lambda - \delta, \lambda + \delta)) = x((\lambda - \epsilon, \lambda + \epsilon))$, then for all $\Delta = (\lambda - \delta, \lambda + \delta) \backslash \{\lambda\}$, $x(\Delta) = \hat{\mathbf{0}}$, and therefore $\Delta \subseteq r(x)$. Thus, any point $\alpha \in \mathbb{R}$ such that $|\alpha - \lambda| < \epsilon$ belongs to the resolvent $r(x)$ and α is an isolated point in the spectrum of $x \in X$. ■

Because $\mathscr{L} = \mathscr{L}(\mathbb{H})$, we can as usual define a function of *dimension* on the finite subspaces of $\mathbb{H}$:

$$\dim\colon \quad \mathscr{E}^\circ \subseteq \mathscr{L}(\mathbb{H}) \mapsto \mathbb{N} \tag{4.8}$$

where $\mathscr{E}^\circ = \{a \in \mathscr{L}(\mathbb{H}) | \exists m_0 \in \mathbb{N} \dim a < m_0\}$.

Consequence 1. Let $x \in X$, and $\exists \lambda \in s_p(x)$, $\exists E \in \mathscr{B}(\mathbb{R})$, such that $\lambda \in E$, $x(E) \in \mathscr{E}^\circ$, and $\forall E' \subseteq E$, such that $\lambda \in E'$, $x(E) = x(E')$. Then

$$\forall E' \subseteq E \dim x(E) = \dim x(E') \tag{4.9}$$

Definition 1. We shall call $\lambda \in s_p(x)$ a *simple discrete point* of the observable x if $(\forall \delta \in \mathbb{R}^+)$ $(\exists \epsilon \in \mathbb{R}^+)$

$$\delta < \epsilon \rightarrow \dim(x((\lambda - \delta, \lambda + \delta))) = 1 \tag{4.10}$$

$\lambda \in s_p(x)$ is called a *discrete point of finite multiplicity* of the observable x if $(\forall \delta \in \mathbb{R}^+)$ $(\exists \epsilon \in \mathbb{R}^+)$

$$\delta < \epsilon \rightarrow \dim(x((\lambda - \delta, \lambda + \delta))) \in \mathbb{N} \tag{4.11}$$

Lemma 2. If $\lambda \in s_p(x)$ is a simple discrete point of the observable $x \in \mathscr{X}$, then there exists a state m_λ such that $(\forall \delta \in \mathbb{R}^+)$ $(\exists \epsilon \in \mathbb{R}^+)$

$$\delta < \epsilon \rightarrow m_\lambda(x((\lambda - \delta, \lambda + \delta))) = 1 \tag{4.12}$$

& $\forall E \in \mathscr{B}(\mathbb{R})$

$$\lambda \notin E \rightarrow m_\lambda(x(E)) = 0 \tag{4.13}$$

Proof. It is sufficient to choose $\phi_\lambda \in \mathbb{H}$, $\|\phi_\lambda\| = 1$ such that

$$P^{x((\lambda - \delta, \lambda + \delta))}\phi_\lambda = \phi_\lambda \qquad \text{if} \quad \delta < \lambda \tag{4.14}$$

It is possible in a unique way, since

$$\dim P^{x((\lambda - \delta, \lambda + \delta))} = 1 \qquad \text{if} \quad \delta < \lambda \tag{4.15}$$

The required state is

$$m_\lambda(M) = (\phi_\lambda, P^M \phi_\lambda), \qquad M \in \mathscr{L}(\mathbb{H}) \tag{4.16}$$

Let $E \in \mathscr{B}(\mathbb{R})$ such that $(\lambda - \delta, \lambda + \delta) \cap E = \varnothing$. Then

$$(\phi_\lambda, P^{x(E)}\phi_\lambda) = (\phi_\lambda, P^{x((\lambda - \delta, \lambda + \delta))} P^{x(E)} \phi_\lambda) \tag{4.17}$$

But by the properties of $\mathscr{L}$-valued measures, if

$$x(E) \perp x((\lambda - \delta, \lambda + \delta)) \tag{4.18}$$

then $P^{x(E)} P^{x((\lambda - \delta, \lambda + \delta))} = 0$. Thus $m_\lambda(E) = 0$. ■

The physical meaning of the state m_λ is that in this state the probability to find the system in any sufficiently small interval of the values of $x \in \mathscr{X}$ containing λ is one with certainty, and in any other interval it is zero.

We see that if $\{\lambda_i | i \in \mathbb{N}\} \subseteq s(x)$ is a family of isolated points in the spectrum of $x \in \mathscr{X}$, then there exists a set of states $\{m_{\lambda_i} | i \in \mathbb{N}\}$ parametrized by the points of the spectrum of $s(x)$, and such that $m_{\lambda_i}(x(\lambda_j)) = \delta_{ij}$[5].

Definition 2. The states m_λ of the previous lemma we shall call the *eigenstates* of the observables $x \in \mathscr{X}$.

If we remember that $x(E)$ belongs to Hilbert space $\mathbb{H}$, we get the following result.

Conclusion 6. If m_λ and $m_{\lambda'}$ are the eigenstates corresponding to the simple discrete eigenvalues λ and λ' of the observable $x \in \mathscr{X}$, then

1. Gleason's pure states m_ϕ and $m_{\phi'}$ correspond accordingly to these states.
2. The corresponding normalized vectors ϕ and ϕ' are orthogonal in $\mathbb{H}$.

The *mean value* of the observable $x \in X$ in the state m is (concerning only the discrete part of the spectrum)

$$m(x) = \sum_{\lambda \in s_p(x)} \lambda m(x(\lambda)) \tag{4.20}$$

Thus, if m_λ is an eigenstate of the observable x, corresponding to the eigenvalue $\lambda \in s_p(x)$, then clearly $m_\lambda(x) = \lambda$.

4.2. Observables with Continuous Spectrum

Lemma 3. If $E \in \mathscr{B}(\mathbb{R})$ and $E \in s_c(x)$ for $x \in \mathscr{X}$, then $\forall A, B \subset E$ such that $A, B \in \mathscr{B}(\mathbb{R})$,

$$A \subset B \rightarrow x(A) < x(B) \tag{4.21}$$

Proof. Let $B = A \perp A_\epsilon$, $A \cap A_\epsilon = \varnothing$. Then $x(A \cup A_\epsilon) = x(A) \vee x(A_\epsilon)$. Since $A_\epsilon \subset E$, then $x(A_\epsilon) \neq \mathbf{0}$ by the spectrum definition. Also by definition $x(A) \perp (A_\epsilon)$; consequently, $\neg(x(A_\epsilon) \leq x(A))$, and $x(A) < x(B)$. ■

Lemma 4. If $E \in \mathscr{B}(\mathbb{R})$, $E \subseteq s_c(x)$ for the observable $x \in \mathscr{X}$, then $\dim(x(E')) > m$, $\forall E' \subseteq E$ and $\forall m \in \mathbb{N}$ [dimension of the subspace $x(E')$ in $\mathbb{H}$ cannot be a finite number].

Proof. From the contrary. Let $\exists E' \subset E$ and $\exists m_0 \in \mathbb{N}$ such that

$$\dim(x(E')) = m_0 \tag{4.22}$$

[5]Here we denoted by $x(\lambda)$ the value of the $\mathscr{L}$-valued measure x such that $\exists \epsilon \in \mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+$,

$$\delta < \epsilon \rightarrow x(\lambda) = x((\lambda - \delta, \lambda + \delta)) \tag{4.19}$$

Then $\forall E'' \subset E'$, $x(E'') < x(E')$, and $\dim(x(E'')) < \dim(x(E'))$. It is true because of the previous lemma, and because of the ordering of subspaces of $\mathbb{H}$ by inclusion. Since the function dim: $\mathscr{E}^{\circ} \mapsto \mathbb{N}$ is bounded below, one has $\exists m' \in \mathbb{N}$, $\exists I \in \mathscr{B}(\mathbb{R})$, such that $\dim(x(I)) = m'$, and $\forall I' \subset I$

$$\begin{cases} \dim(x(I)) = \dim(x(I)) = m' \\ \quad x(I') < x(I) \end{cases} \tag{4.23}$$

This clearly contradicts the properties of $\mathscr{L}$-valued measures. ■

Now we are ready to show that the description of "eigenstates" corresponding to the points of the continuous spectrum of x has an intrinsic contradiction.

We can in principle define such an "eigenstate" for point λ as

$$(\exists r_0 \in \mathbb{R}^+)\ (\forall r < r_0)\ (m_\lambda(x(B_r(\lambda)) = 1\ \&\ (\forall E \subseteq \mathscr{B}(\mathbb{R}))$$

$$(\lambda \notin E \rightarrow m_\lambda(x(E)) = 0) \tag{4.24}$$

Assume that such a state exists if $\lambda \in s_c(x)$. Thus, it is clear that it is "localized" in the point λ and by the Gleason theorem a one-dimensional subspace of $\mathbb{H}$ corresponds to it. On the other hand, if $B_r(\lambda)$ is a vicinity of $\lambda \in s_c(x)$, then for all $r \in \mathbb{R}^+$ the subspaces $x(B_r(\lambda))$ of $\mathbb{H}$ are *infinite dimensional*.

Thus, one is forced to define a "limit transition" from the infinite dimensional "eigensubspace" corresponding to any infinitely small interval including λ to a one-dimensional eigensubspace corresponding to the *point* λ. Unfortunately, in usual mathematics one cannot proceed in that way, simply because one cannot convert infinity to a finite number.

Moreover, if one defines artificially such an eigenstate for any point $\lambda \in s_c(x)$, the number of such states is obviously uncountable. Again by the Gleason theorem, to any such state a vector $\phi_\lambda \in \mathbb{H}$, $\|\phi_\lambda\| = 1$, coincides, and it is easy to show the mutual orthogonality of these vectors.

Consequently, the existence of an uncountable basis has to be allowed in $\mathbb{H}$, which contradicts the separability of Hilbert space $\mathbb{H}$.

Conclusion 7. 1. In the formalism of quantum mechanics based on the quantum logic $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \mathscr{G} \rangle$, eigenstates associated with the unit vectors of Hilbert space $\mathbb{H}$ are defined only for isolated points of the observable spectra.

2. An element of $\mathscr{L}(\mathbb{H})$ coinciding with a point of the spectrum of an observable $x \in \mathscr{X}$ and defined as

$$x(\lambda) = x((\lambda - \delta, \lambda + \delta)) \in \mathscr{L}(\mathbb{H}) \tag{4.25}$$

for any sufficiently small δ, exists only for isolated points of $s(x)$. Its existence is the necessary and sufficient condition for $\lambda \in \mathbb{R}$ to belong to the discrete spectrum of the observable x.

3. The Dirac formalism in which the eigenstates exist for *all points* of the observables spectrum cannot follow from the quantum logic $\langle \mathscr{X}, \mathscr{L}(\mathbb{H}), \mathscr{G} \rangle$.

5. NONSTANDARD QUANTUM LOGIC

In this section we shall build a nonstandard extension of the quantum logic formulated in the previous section.

We suppose that the following sets are contained in the set of individuals S of the standard superstructure $\hat{V}(S)$:

$\mathbb{R}$—complete ordered field of real numbers
$\mathscr{L}$—full irreducible OAC-lattice of propositions
$\mathbb{C}$—field of complex numbers
$\mathbb{H}$—separable Hilbert space over $\mathbb{C}$

Then the family of Borel subsets $\mathscr{B}(\mathbb{R})$, the family of $\mathscr{L}$-valued measures $\mathscr{X}$, the lattice $\mathscr{L}(\mathbb{H})$, and the family of probability measure $\mathscr{G}$ are contained in the standard superstructure $\hat{V}(S)$.

The set of nonstandard individuals $^*S$ is then

$$^*S = {}^*\mathbb{R} \cup {}^*\mathscr{L} \cup {}^*\mathbb{C} \cup {}^*\mathbb{H} \tag{5.1}$$

The fields $^*\mathbb{R}$, $^*\mathbb{C}$, and the nonstandard Hilbert space are considered in Davis (1977), Stroyan and Luxemburg (1976), Albeverio *et al.* (1950), and Farrukh (1975). We are left with the extended lattice $^*\mathscr{L}$.

The study of $^*\mathscr{X}$, $\mathscr{B}(^*\mathbb{H})$, $^*\mathscr{L}(\mathbb{H})$, and $^*\mathscr{G}$ proceeds by the transfer theorem (Theorem 1) because these sets are defined in the standard superstructure by axiom systems (formulas) of the language $L(\hat{V}(S))$.

Accordingly, these axioms are true also by their interpretation in the nonstandard superstructure $\hat{V}(^*S)$ if one adds the adjective "internal" where it is needed. Thus a nonstandard (extended) quantum logic can also be called an "internal quantum logic."

5.1. Lattice of Propositions

We suppose that $\mathscr{L} \subseteq S$. This means that the nonstandard extension of $\mathscr{L}$ can be built by the ultraproduct construction (2.3). If I is some index set, and $\mathscr{F}$ is a free ultrafilter on I, then $^*\mathscr{L} = \mathscr{L}^I/\mathscr{F}$ is a quotient set of the set of functions from I to $\mathscr{L}$ by the equivalence relation

$$(f \sim g \rightarrow \{i \in I | f(i) = g(i)\} \in \mathscr{F}, \qquad f, g \in \mathscr{L}^I \tag{5.2}$$

The nontriviality of such an extension, i.e., $\mathscr{L} \subseteq {}^*\mathscr{L}$ and $\mathscr{L} \neq {}^*\mathscr{L}$, depends on the choice of I and $\mathscr{F}$. But its properties under consideration do not depend on this choice providing nontriviality of the extension of the real field $\mathbb{R}$. This is obviously so in our model, thus we shall not make I and $\mathscr{F}$ concrete.

Applying the transfer theorem to the lattice $\mathscr{L}$ defined by the axiom system of Section 3, one gets the following result.

Theorem 3. Let $\mathscr{L} \subseteq S$ and Axioms 6–9 be true in $\hat{V}(S)$.

Then ${}^*\mathscr{L}$ is an irreducible OAC-lattice, where every internal subset of it has the least upper and the greatest lower bounds.

Proof. Trivial usage of the transfer theorem. For example, sentence 1 of Axiom 6 is translated to the following one:

$$\forall x \in {}^*\mathscr{L}\ (x \vee x = x)\ \&\ (x \wedge x = x) \tag{5.3}$$

Internal operations ${}^*\vee$ and ${}^*\wedge$ are continuations of the operations $\vee$ and $\wedge$ on $\mathscr{L}$, because $\mathscr{L}$ is contained in the set of individuals S. Thus we shall omit stars at these operations.

Completeness of the lattice $\mathscr{L}$ is translated into $(\forall A \in {}^*\mathscr{P}(\mathscr{L}))$ $(\exists a \in {}^*\mathscr{L})$ $(\forall x \in A)$

$$x \wedge a = a \tag{5.4}$$

& $(\exists t \in {}^*\mathscr{L})$ $(\forall x \in A)$

$$x \wedge t = t \rightarrow a \wedge t = t \tag{5.5}$$

and dually for $\vee$.

Thus the notions of the least upper bound and the greatest lower bound cannot be defined for every subset of ${}^*\mathscr{L}$, but for internal sets of ${}^*\mathscr{L}$ only. ■

Definition 3. A lattice in the nonstandard superstructure $\hat{V}({}^*S)$ where every internal subset of it [such that $A \subseteq {}^*\mathscr{L}$ and $A \in {}^*\hat{V}(S)$] has the least upper bound and greatest lower bound is called *internally complete.*

Then ${}^*\mathscr{L}$ is internally complete.

The universal boundaries of $\mathscr{L}$ and ${}^*\mathscr{L}$ coincide. Since $\mathscr{L} \subseteq S$, $\hat{\mathbf{1}} \in \mathscr{L}$, $\hat{\mathbf{0}} \in \mathscr{L}$, then ${}^*\hat{\mathbf{0}} = \hat{\mathbf{0}}$, ${}^*\hat{\mathbf{1}} = \hat{\mathbf{1}}$, i.e., the universal boundaries are standard elements of ${}^*\mathscr{L}$. Because of the property

$$\forall x \in \mathscr{L}\ (\hat{\mathbf{0}} \leq x)\ \&\ (x < \hat{\mathbf{1}}) \tag{5.6}$$

in $\mathscr{L}$, one has in ${}^*\mathscr{L}$

$$\forall x \in {}^*\mathscr{L}\ (\hat{\mathbf{0}} \leq x)\ \&\ (x \leq \hat{\mathbf{1}}) \tag{5.7}$$

This proves that $\hat{\mathbf{0}}$ and $\hat{\mathbf{1}}$ actually are the universal boundaries in ${}^*\mathscr{L}$.

A more detailed study of $^*\mathscr{L}$ is not given here. We turn now to the lattice of subspaces of Hilbert space $\mathscr{L}(\mathbb{H})$. The existence of the isomorphism μ between $\mathscr{L}$ and $\mathscr{L}(\mathbb{H})$ (Conclusion 5) leads to the following statement.

Theorem 4. $^*\mathscr{L}(\mathbb{H})$ is an internally complete OAC-lattice. Elements of $^*\mathscr{L}(\mathbb{H})$ are closed internal subsets of nonstandard Hilbert space $^*\mathbb{H}$.

The proof of this theorem is a straightforward application of the transfer theorem to the μ mapping. Since the Hilbert space $\mathbb{H}$ and the lattice of propositions $\mathscr{L}$ together belong to the set of individuals, the bijection μ belongs to the standard universum $\hat{V}(S)$. But because the elements of the set of all the closed subspaces of $\mathbb{H}$, $\mathscr{L}(\mathbb{H})$, do not lie in the set of individuals, the extension of the mapping μ in $\hat{V}(^*S)$ will not be defined on all the elements of $\mathscr{L}(^*\mathbb{H})$ (the set of all closed subspaces of $^*\mathbb{H}$), but on the internal sets of it only [belonging to $^*\mathscr{L}(\mathbb{H})$].

5.2. Range of Observables

The family of the Borel subsets $\mathscr{B}(\mathbb{R})$ in the standard universum $\hat{V}(S)$ is defined by Axiom 1. By the transfer theorem, one gets the definition of $^*\mathscr{B}(\mathbb{R})$ in the nonstandard superstructure:

1. $\forall\{X_i | i\in\mathbb{N}\}\in{}^*\mathscr{P}(\mathscr{B}(\mathbb{R}))$

$$\left(\bigcup_{i\in{}^*\mathbb{N}} X_i\in{}^*\mathscr{B}(\mathbb{R})\right) \;\&\; \left(\bigcap_{i\in{}^*\mathbb{N}} X_i\in{}^*\mathscr{B}(\mathbb{R})\right) \tag{5.8}$$

2. $(\forall E\in{}^*\mathscr{B}(\mathbb{R}))\ ({}^*\mathbb{R}\backslash E\in{}^*\mathscr{B}(\mathbb{R}))$.
3. $(\forall x\in{}^*\mathbb{R})\ (\forall r\in{}^*\mathbb{R}^+)$

$$B_r(x) = \{y\in{}^*\mathbb{R} |\ |x-y| < r\}\in{}^*\mathscr{B}(\mathbb{R}) \tag{5.9}$$

In other words, $^*\mathscr{B}(\mathbb{R})$ is a family of internal subsets of hyperreal axis $^*\mathbb{R}$, which contains unions and intersections of any internal hypersequences (parametrized by the nonstandard natural numbers $^*\mathbb{N}$), complementations to $E\in{}^*\mathscr{B}(\mathbb{R})$, and internal open balls in $^*\mathbb{R}$.

5.3. Observables

The set of observables in the standard quantum logic is the family of $\mathscr{L}$-valued measures $\mathscr{X}$ (Conclusion 4).

By the transfer theorem we can get the set of *internal observables* $^*\mathscr{X}$. If $x\in{}^*\mathscr{X}$, then $x\in{}^*\mathscr{L}(\mathbb{H})\times{}^*\mathscr{B}(\mathbb{R})$, x maps $^*\mathscr{B}(\mathbb{R})$ to $^*\mathscr{L}(\mathbb{H})$, and

$^*\mathscr{X} \in {}^*\mathscr{P}({}^*\mathscr{L}(\mathbb{H}) \times {}^*\mathscr{B}(\mathbb{R}))$ such that:

1. $(\forall E_1 \in {}^*\mathscr{B}(\mathbb{R}))\ (\forall E_2 \in {}^*\mathscr{B}(\mathbb{R}))\ (\forall \mu \in {}^*\mathscr{X})$

$$E_1 \cap E_2 = \varnothing \rightarrow \mu(E_1) \perp \mu(E_2) \tag{5.10}$$

2. $\forall \{E_i \mid i \in {}^*\mathbb{N}\} \in {}^*\mathscr{P}(\mathscr{B}(\mathbb{R}))\ (i \neq j \rightarrow E_i \cap E_j = \varnothing)\ (\forall \mu \in {}^*\mathscr{X})$

$$\mu\left(\bigcup_{i \in {}^*\mathbb{N}} E_i\right) = \bigvee_{i \in {}^*\mathbb{N}} \mu(E_i) \tag{5.11}$$

3. $(\forall \mu \in {}^*\mathscr{X})$

$$(\mu(\varnothing) = \hat{0})\ \&\ (\mu({}^*\mathbb{R}) = \hat{1}) \tag{5.12}$$

Because of the isomorphism between the elements of lattice $\mathscr{L}(\mathbb{H})$ and the orthogonal projectors on the closed subspaces of $\mathbb{H}$, we can say that an internal projector on the closed internal subspace of $^*\mathbb{H}$ corresponds to every element of $^*\mathbb{H}$. Since a projector-valued measure $P_E = P^{x(E)}$, $E \in \mathscr{B}(\mathbb{R})$, corresponds to every observable $x \in \mathscr{X}$, we put into coincidence to every $x \in {}^*\mathscr{X}$, an internal projector-valued "measure" $P_E = P^{x(E)}$, $E \in {}^*\mathscr{B}(\mathbb{R})$, with the properties: $(\forall x \in {}^*\mathscr{X})$

1. $\forall E_1 \in {}^*\mathscr{B}(\mathbb{R}),\ \forall E_2 \in {}^*\mathscr{B}(\mathbb{R})$

$$E_1 \cap E_2 = \varnothing \rightarrow P_{E_1} P_{E_2} = P_{E_2} P_{E_1} = 0 \tag{5.13}$$

2. $\forall \{E_i \mid i \in {}^*\mathbb{N}\} \in {}^*\mathscr{P}(\mathscr{B}(\mathbb{R}))\ (i \neq j \rightarrow E_i \cap E_j = \varnothing)$

$$P_{\bigcup E_i} = \sum_{i \in {}^*\mathbb{N}} P_{E_i} \tag{5.14}$$

3.

$$P_{\varnothing} = 0, \qquad P_{{}^*\mathbb{R}} = 1 \tag{5.15}$$

Such a mapping is not a measure in the usual sense, because the hyperreal sequences are parametrized by the numbers of $^*\mathbb{N}$ and are not countable.

5.4. States

States in the standard theory are functions from $\mathscr{L}(\mathbb{H})$ to $[0, 1] \subseteq \mathbb{R}$ with the properties of probability measures (Conclusion 3).

Exactly as was done for the observables, one can build the internal set of internal functions $^*\mathscr{G} \in {}^*\hat{V}(S)$.

Then $^*\mathscr{G}$ is an order-defining family of $^*\sigma$-convex *-probability "measures" on $^*\mathscr{L}(\mathbb{H})$.

More clearly, $*\sigma$-convexity is an internal analog of σ-convexity in the nonstandard case:

$(\forall\{\hat{\alpha}_i(\cdot)|i\in{}^*\mathbb{N}\}\in{}^*\mathscr{P}(\mathscr{G}))$ $(\forall\{t_i|i\in{}^*\mathbb{N}\}\in{}^*\mathscr{P}(\mathbb{R}^+))$ $(\sum_{i\in{}^*\mathbb{N}}t_i=1)$ $(\exists\hat{\alpha}(\cdot)\in{}^*\mathscr{G})$

$$\left(\hat{\alpha}=\sum_{i\in{}^*\mathbb{N}} t_i\hat{\alpha}_i\right) \tag{5.16}$$

i.e., internal $*\sigma$-convex hypersums of the elements of $*\mathscr{G}$ also are in $*\mathscr{G}$.

The sentence "the elements of $*\mathscr{G}$ are $*$-probability measures on $*\mathscr{L}(\mathbb{H})$" means:

The value of a measure $m\in{}^*\mathscr{G}$ on the element of $*\mathscr{L}$ which is the least upper bound of an internal hypersequence of orthogonal elements of $*\mathscr{L}$ is equal to the internal hypersum of the values of the measure m on the elements of this hypersequence:

$(\forall\hat{\alpha}(\cdot)\in{}^*\mathscr{G})$ $(\forall\{\{a_i\}_{i\in{}^*\mathbb{N}}\}\in{}^*\mathscr{P}(\mathscr{L}(\mathbb{H})))$ $(i\neq j\rightarrow a_i\perp a_j)$

$$\hat{\alpha}\left(\bigvee_{i\in{}^*\mathbb{N}} a_i\right)=\sum_{i\in{}^*\mathbb{N}}\hat{\alpha}(a_i) \tag{5.17}$$

It is clear that the measure m is not a usual probability measure.

By the Gleason theorem, any element $m\in\mathscr{G}$ is of the form $m(\cdot)=\sum_{i\in\mathbb{N}}t_i(\phi_i, P^{(\cdot)}\phi_i)$.

The nonstandard analog of this is

$$m\in{}^*\mathscr{G}\rightarrow m(\cdot)=\sum_{i\in{}^*\mathbb{N}} t_i(\phi_i, P^{(\cdot)}\phi_i) \tag{5.18}$$

where t_i is an internal hypersequence $\{t_i|i\in{}^*\mathbb{N}\}\in{}^*\mathscr{P}(\mathbb{R}^+)$, $\sum_{i\in{}^*\mathbb{N}}t_i=1$, and ϕ_i is an internal hypersequence of vectors of $*\mathbb{H}$:

$$\{\phi_i|i\in{}^*\mathbb{N}\}\in{}^*\mathscr{P}(\mathbb{H}),\ \|\phi_i\|=1, \qquad \forall i\in{}^*\mathbb{N} \tag{5.19}$$

Note that the space $*\mathbb{H}$ cannot be separable, because of the nontriviality of our extension. Then an uncountable orthogonal basis can exist in $*\mathbb{H}$. Thus, as will be shown later, a set of eigenstates parametrized by the points of the continuous spectrum of some internal observable $x\in{}^*\mathscr{X}$ can exist in the nonstandard quantum logic.

6. SPECTRAL AXIOM AND DIRAC'S FORMALISM

Here we shall discuss spectral (more exactly, hyperspectral) properties of internal observables $x\in{}^*\mathscr{X}$.

Since the hyperspectrum of the internal observables from $^*\mathscr{X}$ is defined by the transfer principle, it is a subset of the hyperreal axis $^*\mathbb{R}$. But if we suppose that its finite part [lying in the set of finite numbers of $^*\mathbb{R}$, see (2.8)] is a subset of the usual standard numbers $^*\mathbb{R}$, then the extended quantum logic will have properties of the Dirac bra-ket formalism. That is, in this quantum theory the spectral properties of the observables are the same for both those with discrete and with continuous spectrum, because they all will have the *hyperdiscrete* spectrum according to $^*\mathbb{R}$. The special condition for this to be true will be introduced in our Spectral Axiom.

The physical meaning of this axiom can be interpreted as follows:

"Propositions" which correspond to the internal observables $x(E)$ and $x(E')$ from $^*\mathscr{X}$ are supposed to be equal if the intervals E and E' from $^*\mathscr{B}(\mathbb{R})$ are different only on an infinitesimal set $\Delta = (E\backslash E') \cup (E'\backslash E)$ such that $x, y \in \Delta \to x \approx y$. In other words, if we call Δ the error of measuring an observable $x \in {}^*\mathscr{X}$, then the infinitesimal measuring error does not lead to a change of proposition. Thus, such an error does not influence the probability value to find the physical system in the interval E of the range of the observable $x \in {}^*\mathscr{X}$.

This in principle corresponds with our intuition on the nature of measuring apparatuses. Actually, any measuring apparatus can distinguish only the finite values of observables and with a definite nonzero error. If we formulate this correctly in the language of infinitesimal errors, we shall get a formalism of quantum mechanics with Dirac's properties.

6.1. Hyperresolvent and Hyperspectrum

The hyperspectrum of an internal observable $x \in {}^*\mathscr{X}$ is a subset of the hyperreal axis $^*\mathbb{R}$, which we shall define by the transfer principle. Remembering the definitions of the standard resolvent and spectrum [see (4.1), (4.2)], in the nonstandard universum we have $(\forall x \in {}^*\mathscr{X})$

$$^*r(x) = \bigcup \{I \in {}^*\mathscr{B}(\mathbb{R}) \mid \alpha(x(I)) = 0 \ \forall \alpha \in {}^*\mathscr{G}\} \ \& \ ({}^*s(x) = {}^*\mathbb{R}\backslash {}^*r(x)) \quad (6.1)$$

We shall call $^*r(x)$ the *hyperresolvent* and $^*s(x)$ the *hyperspectrum* of an internal observable $x \in {}^*\mathscr{X}$.

Since the spectrum is defined by the formula of $L(\hat{V}(S))$, the hyperspectrum $^*r(x)$ of $x \in {}^*\mathscr{X}$ in $\hat{V}({}^*S)$ is an internal set.

We shall introduce now the promised spectral axiom about the internal observables $x \in {}^*\mathscr{X}$.

Axiom 12. Spectral Axiom. The intersection between the hyperspectrum $^*s(x)$ of internal observables and the finite part of the hyperreal axis

Fin($^*\mathbb{R}$) is a subset of the set of standard numbers $\mathbb{R}$[6]

$$\forall x \in {}^*\mathscr{X}\ ({}^*r(x) \cap \mathrm{Fin}({}^*\mathbb{R}) \subseteq \mathbb{R}) \tag{6.2}$$

Definition 4. The hyperspectrum $^*s(x)$ of an internal observable $x \in {}^*\mathscr{X}$ is called *hyperdiscrete* in a point $\lambda \in \mathbb{R}$ if: $\exists \epsilon \in {}^*\mathbb{R}^+$, $\forall \delta \in {}^*\mathbb{R}^+$,

$$\delta < \epsilon \rightarrow x(B_\delta(\lambda)) = x(B_\epsilon(\lambda)) \tag{6.3}$$

This definition is exactly transferred into the nonstandard universum condition (4.4) for the spectrum to be discrete.

6.2. Spectral Properties of Internal Observables

We shall define as $^*\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle_s$ the nonstandard quantum logic such that the Spectral Axiom 12 is true in it.

Definition 5. We shall call by the *spectrum* of the internal observable $x \in {}^*\mathscr{X}$ in $^*\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle_s$ theory the finite (and, consequently, standard) part of its hyperspectrum

Theorem 5. Every observable $x \in {}^*\mathscr{X}$ in $^*\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle_s$ theory has hyperdiscrete spectrum.

Proof. If $I \in {}^*\mathscr{B}(\mathbb{R})$, $I \cap \mathbb{R} = \varnothing$, then by Axiom 12 and by the definition (6.1) of hyperspectrum, $\forall \alpha \in {}^*\mathscr{X}\ (\alpha(x(I)) = 0)$.

Let $\lambda \in {}^*s(x)$, $\lambda \in \mathbb{R}$, and $B_{r'}(\lambda) \subset B_r(\lambda)$, where $B_r(\lambda)$ is an internal ball in $^*\mathbb{R}$ with its center in the standard point λ.

If $r \approx 0$, $r' \approx 0$, $r \neq r'$, then

$$\Delta = B_r(\lambda) \backslash B_{r'}(\lambda)\ \&\ \Delta \nsubseteq \mathbb{R} \tag{6.4}$$

and Δ is an internal set.

Then $x(B_r(\lambda)) = x(B_{r'}(\lambda) \cup \Delta) = x(B_{r'}(\lambda) \vee x(\Delta)) = x(B_{r'}(\lambda))$, since $\Delta \subseteq$ Fin(*R) and $\Delta \nsubseteq \mathbb{R} \rightarrow x(\Delta) = \hat{0}$. Thus, for any point $\lambda \in \mathbb{R}$ of the spectrum of $x \in {}^*\mathscr{X}$ there exists its infinitesimal vicinity such that the values of $^*\mathscr{L}$-valued measure $x(\cdot)$ are constant within it. ■

Now we shall use the following consequence of the fact that the real axis $\mathbb{R}$ possesses the Hausdorff topology (Albeverio *et al.*, 1986): there is a one-to-one correspondence between every standard point λ of $\mathbb{R}$ and a set

[6]It can be shown also that the hyperspectrum must contain also some infinite points from $^*\mathbb{R}$ if it contains all the real axis $^*\mathbb{R}$ (Albeverio *et al.*, 1986). In other words, if the hyperspectrum of $x \in {}^*\mathscr{X}$ is "unbounded" in the usual sense, it contains also the "limit" infinite points from Inf($^*\mathbb{R}$).

of points infinitesimally close to it (the "monad" of this point)

$$\mu(\lambda) = \{y \in {}^*\mathbb{R} |\ |\lambda - y| \approx 0\} \tag{6.5}$$

Then to all the propositions $x(B_r(\lambda))$, $r \approx 0$, $\lambda \in \mathbb{R}$, defined in ${}^*\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle_s$ theory one can put to coincidence the proposition $x(\lambda)$ for any $\lambda \in \mathbb{R}$ as

$$x(\lambda) = x(B_{r \approx 0})(\lambda) \tag{6.6}$$

This is so since the ball $B_r(\lambda)$, $r \approx 0$, belongs to the monad of the point λ, and by theorem 5 the proposition $x(B_r(\lambda))$ is not changed for any $r \approx 0$.

The function of dimension (4.8), dim: $\mathscr{E}^\circ \subseteq \mathscr{L}(\mathbb{H}) \mapsto \mathbb{N}$, can be extended in ${}^*\hat{V}(S)$:

$$^*\text{dim:} \quad {}^*\mathscr{E}^\circ \subseteq {}^*\mathscr{L}(\mathbb{H}) \mapsto {}^*\mathbb{N} \tag{6.7}$$

The *dim function is defined on the hyperfinite (Davis, 1977) subspaces of ${}^*\mathscr{L}(\mathbb{H})$, i.e., such that

$$a \in {}^*\mathscr{E}^\circ \leftrightarrow \exists n \in {}^*\mathbb{N}\ {}^*\text{dim}(a) < n \tag{6.8}$$

If ${}^*\text{dim}(x(E)) \in \mathbb{N}$, we say that the dimension of the subspace $x(E) \in {}^*\mathscr{L}(\mathbb{H})$ is *finite*. If ${}^*\text{dim}(x(E)) \in {}^*\mathbb{N} \backslash \mathbb{N}$, it is *hyperfinite*.

Definition 6. 1. If ${}^*\text{dim}(x(\lambda)) = 1$, then λ is a simple hyperdiscrete point of the spectrum of the internal observable x.

2. If ${}^*\text{dim}(x(\lambda)) \in \mathbb{N}$, then λ is a point of the spectrum of x of finite multiplicity.

3. If ${}^*\text{dim}(x(\lambda)) \in {}^*\mathbb{N} \backslash \mathbb{N}$, then λ is a point of hyperfinite (or infinite) multiplicity of the spectrum of the observable $x \in {}^*\mathscr{X}$.

Note that we have given such definitions for the points of the *spectrum* (the finite part of the hyperspectrum) of the internal observables $x \in {}^*\mathscr{X}$ because the propositions $x(\lambda)$ are defined only for the finite points of the hyperreal axis ${}^*\mathbb{R}$.

6.3. Eigenstates

To every simple hyperdiscrete point of the spectrum of an observable $x \in {}^*\mathscr{X}$ of ${}^*\langle \mathscr{X}, \mathscr{L}, \mathscr{G} \rangle_s$ theory an eigenstate corresponds with the properties analogous to those of the standard quantum logic (4.12), (4.13).

Transferring the Gleason theorem to the nonstandard universum, one has: for every pure state from ${}^*\mathscr{G}$ there exists one and only one internal function m_ϕ such that

$$m_\phi(M) = (\phi, P^M \phi), \qquad \phi \in {}^*\mathbb{H}, \quad \|\phi\| = 1, \quad M \in {}^*\mathscr{L}(\mathbb{H}) \tag{6.9}$$

The set of states $^*\mathscr{G}$ is nothing else but the family of all convex combinations of pure states like m_ϕ.

Then, let $m = m_\phi$, $\phi \in {}^*\mathbb{H}$, $\|\phi\| = 1$, $\lambda \in \mathbb{R}$, and

$$P^{x((\lambda-\delta,\lambda+\delta))}\phi = \phi \qquad \text{if} \quad \delta \approx 0 \tag{6.10}$$

Such a choice is possible because $^*\dim(x((\lambda-\delta, \lambda+\delta))) = 1$ by Definition 6.

It can be easily shown also that

$$P^{x((\lambda-\delta,\lambda+\delta))}P^{x((\lambda'-\delta',\lambda'+\delta'))} = 0, \qquad \lambda \neq \lambda', \quad \delta \approx 0, \quad \delta' \approx 0 \tag{6.11}$$

Indeed:

1. The metric topology of the real axis is the Hausdorff one. Thus, if $\delta \approx 0$, $\delta' \approx 0$, and $\lambda \neq \lambda'$, $\lambda, \lambda' \in \mathbb{R}$, then
$$(\lambda-\delta, \lambda+\delta) \cap (\lambda'-\delta', \lambda'+\delta') = \varnothing \tag{6.12}$$
2. Properties of the projector-valued measures [see (3.39)–(3.41)] are translated to the nonstandard universum.
3. The interval $(\lambda-\delta, \lambda+\delta)$ lies in $^*\mathscr{B}(\mathbb{R})$ and therefore is an internal set.

Then,

$$m_\phi(x((\lambda-\delta, \lambda+\delta))) = (\phi, P^{x((\lambda-\delta,\lambda+\delta))}\phi) = 1$$

and

$$\begin{aligned} m_\phi(x((\lambda'-\delta', \lambda'+\delta'))) &= (\phi, P^{x((\lambda'-\delta',\lambda'+\delta'))}\phi) \\ &= (\phi, P^{x((\lambda-\delta,\lambda+\delta))}P^{x((\lambda'-\delta',\lambda'+\delta'))}\phi) = 0 \end{aligned}$$

for any $\lambda \neq \lambda'$ and $\delta, \delta' \approx 0$.

Thus, the state $m_\phi = m_{\phi_\lambda}$ is an eigenstate by the definition (4.12), (4.13), λ being the corresponding eigenvalue.

We conclude that for every internal observable $x \in {}^*\mathscr{X}$ a set of eigenstates can exist which is parametrized by the points of its spectrum (not hyperspectrum!) and, possibly, not countable:

$$\begin{gathered} \{m_{\phi_\lambda} | \lambda \in \mathbb{R}\} \subseteq {}^*\mathscr{G} \\ m_{\phi_\lambda}(x(\lambda')) = \delta_{\lambda\lambda'} = \begin{cases} 0, & \lambda \neq \lambda' \\ 1, & \lambda = \lambda' \end{cases} \end{gathered} \tag{6.13}$$

Since the unit vectors $\phi_\lambda \in {}^*\mathbb{H}$ correspond to the states m_{ϕ_λ}, it is easy to see that if $\lambda \neq \lambda'$, then $(\phi_\lambda, \phi_{\lambda'}) = 0$.

Thus, a family of orthogonal vectors $\{\phi_\lambda | \lambda \in \mathbb{R}\} \subseteq {}^*\mathbb{H}$ corresponds to every internal observable $x \in {}^*\mathscr{X}$.

The mean value of an internal observable $x \in {}^*\mathscr{X}$ in the eigenstate $m_{\phi_\lambda} = m_\lambda$ is

$$m_\lambda(x) = {}^*\sum_{\lambda' \in \mathbb{R}} \lambda' m_\lambda(x(\lambda')) = \lambda \tag{6.14}$$

So, this definition is consistent.

7. CONCLUSIONS

The extended quantum logic (which was built by applying the transfer theorem to the usual quantum logic), where the Spectral Axiom 12 is true for all the internal observables, admits the existence of eigenstates for *any* finite points of their spectra.

If an internal observable has a simple hyperdiscrete spectrum, then a family of eigenstates m_λ corresponds to it, and the corresponding vectors in hyper-Hilbert space ${}^*\mathbb{H}$ form a (possibly uncountable and not necessary full) orthonormal basis in ${}^*\mathbb{H}$.

If the physical system is in the eigenstate m_λ of $x \in {}^*\mathscr{X}$, then the mean value of x in this state is λ (which is trivial but nevertheless important for consistency of the new formalism).

Finally, these properties do not depend at all on the kind of observable spectrum, i.e., it can be discrete or continuous in $\mathbb{R}$.

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